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A minimum problem on matrices

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by
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In a previous report (Rapport ZW 1954-003) the question was put to investigate for which L, independent of E, of the expression

(1) $$\tau^2 = E'(LP)^{-1} LCL'(P'L')^{-1} E$$

attains its minimal value,
here

C is a constant positive definite m x m matrix;
L is a variable k x m matrix;
P is a constant m x k matrix of rank k;
E is a constant k x 1 matrix;
$k \leq m$.

We prove that the minimum is equal to $\tau_0{}^2 = E'(P'C^{-1}P)^{-1} E$ and that the minimum is attained if and only if $L = VP'C^{-1}$, where V is an arbitrary non singular k x k matrix.

One has

(2) $$\tau^2 - \tau_0{}^2 = E'(LP)^{-1} LCL'(P'L')^{-1} E - E'(P'C^{-1}P)^{-1} E = $$
$$= E'(LP)^{-1}L \left\{ C - P(P'C^{-1}P)^{-1} P' \right\} L'(P'L')^{-1} E.$$

We first prove the following

Lemma. If C is a positive definite m x m matrix and if A is an m x m matrix subject to the condition

(3) $$A'C^{-1} A = A ,$$

then C-A is a positive semidefinite matrix.
Furthermore an m x 1 matrix U satisfies
$$U'(C-A)U = 0$$
if and only if

(4) $$(C-A)U = 0.$$

Proof. In virtue of (3) one has $A' = A'C^{-1} A = A$ and further
$$U'(C-A)C^{-1}(C-A)U = U'(C-A-A+AC^{-1}A)U = U'(C-A)U.$$
Consequently since C^{-1} is definite, C-A is semidefinite and moreover
$U'(C-A)U = 0$ if and only if (4) holds.

By means of this lemma we can discuss (2) by taking
$$A = P(P'C^{-1}P)^{-1} P'.$$

This m x m matrix satisfies (3) and so

(5) $U'(C - P(P'C^{-1}P)^{-1} P')U \geqq 0$;

in the last relation the equality sign holds if and only if U satisfies

(6) $(C - P(P'C^{-1}P)^{-1} P')U = 0.$

Comparing (2) and (5) we infer $\tau^2 \geqq \tau_0^2$ and $\tau^2 = \tau_0^2$ if (6) holds with

$$U = L'(P'L')^{-1} E.$$

Hence the necessary and sufficient condition for L to minimalise (1) is $CL'(P'L')^{-1} E - P(P'C^{-1}P)^{-1} E = 0.$

This last relation holds identically in E if and only if

$$CL'(P'L')^{-1} = P(P'C^{-1}P)^{-1} ,$$

which -since C and P'L' are non singular- is equivalent to

(7) $L' = C^{-1}P(P'C^{-1}P)^{-1} P'L'.$

From (7) we infer that L' has the form

(8) $L' = C^{-1} PV'$,

where V is a non singular k x k matrix.

Conversely if L' is given by (8) where V is an arbitrary non singular k x k matrix, then obviously L' satisfies (7).

<u>REMARK:</u> In the above argument it appeared that the matrix

(9) $A = P(P'C^{-1}P)^{-1} P'$

(where C is a positive definite m x m matrix and P is an m x k matrix of rank k) satisfies

(10) $A'C^{-1} A = A.$

We now prove that conversely every m x m matrix A of rank $k \leqq m$ which satisfies (10) has the form (9), where P is an m x k matrix the k columns of which are independent linear composita of the columns of A.

In fact let P be an m x k matrix which is subjected to the condition that its k columns are independent linear composita of the columns of A, but further arbitrarily chosen. Then obviously P has the rank k. Now conversely every column of A is a linear combination of the k columns of P, so we have A = PQ', where Q is a suitably chosen m x k matrix , which has the rank k since A is of the rank k. Substitution in (10) gives

$$QP'C^{-1} PQ' = PQ'.$$

Since Q is of rank k this relation is equivalent to

$$QP'C^{-1} P = P.$$

Further since also P is of rank k the k x k matrix $P'C^{-1} P$ has the rank k and therefore is non singular. Consequently

$$Q = P(P'C^{-1} P)^{-1} , \text{ thus } A = P(P'C^{-1} P)^{-1} P'.$$